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Technical Notes

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Stagnation Temperature Effect on the Prandtl Meyer Function

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Nomenclature

a	=	sound velocity
b_1, b_2	=	stretching function coefficients
C_P	=	specific heat to constant pressure
H	=	enthalpy
N	=	discretization points number
R	=	thermodynamic constant of air
T	=	temperature
γ	=	specific heats ratio
ε	=	relative error of computation
ν	=	Prandtl Meyer function

Subscripts

i	=	nodes
S	=	supersonic value
0	=	stagnation condition
*	=	critical condition

I. Introduction

THE Prandtl Meyer (PM) function plays a significant role in the supersonic flows calculation. To design a supersonic nozzle giving a uniform and parallel flow at the exit section [1–3], it is necessary to deviate the nozzle at the throat of an initial expansion angle to have the desired exit Mach number. The design is based on the application of the Method of Characteristics [1,3], which is based on the PM function. A second application for external aerodynamics is to calculate the supersonic flow around a pointed airfoil.

A new form of the PM function is developed like a generalization of the perfect gas (PG) one, by adding the effect of variation of C_P with the temperature. The gas remains perfect; its state's equation remains valid, except it will be calorically imperfect and thermally perfect or gas at high temperature (HT).

The obtained results of a supersonic flow of a perfect gas presented in [4] are valid under the basis of a calorically perfect gas assumption, that is, the specific heats do not depend on the temperature, which is not valid in the real case when T_0 increases.

The new Prandtl Meyer function form is presented by integration of a complex analytical function, where the analytical calculation procedure is impossible. Then, our interest is oriented to the determination of the approximate numerical solutions. The integration is done in an interval containing the temperature T_*, where the function has a high gradient in this point.

The problem encountered in the aeronautical applications is that the use of the designed nozzle on the basis of the PG assumption degrades the desired performances [5] (thrust, exit Mach number), especially if T_0 is high [2,3]. To determine the limit of application of the PG model, a calculation of the error given by this model compared to the HT model is presented.

The table of variation of C_P for air according to T_0 until 3550 K is presented in [6]. A polynomial interpolation with the values of this table is taken to find an analytical form [5,7]. The presented mathematical relations are valid in the general case independently of the interpolation form, but our results are presented by the use a polynomial of ninth degree. The application is for air.

II. Mathematical Formulation

For an infinitesimal oblique shock, the relation between the flow deviation in a point and the speed variation on both sides of the Mach wave is known by the PM function. In the differential form is given by [1]

$$d\nu = (M^2 - 1)^{1/2} \frac{dV}{V} \tag{1}$$

To integrate Eq. (1), the term dV/V and the Mach number must be exprimed by their expressions at high temperature [5]. The following form is obtained:

$$d\nu = -F_\nu(T) \tag{2}$$

where

$$F_\nu(T) = \frac{C_P(T)}{2H(T)} \sqrt{2H(T)/a^2(T) - 1} \tag{3}$$

and [5]

$$M(T) = \frac{\sqrt{2H(T)}}{a(T)} \tag{4}$$

$$a(T) = \sqrt{\gamma(T)RT} \tag{5}$$

$$\gamma(T) = \frac{C_P(T)}{C_P(T) - R} \tag{6}$$

$$H(T) = \int_T^{T_0} C_P(T)\, dT \tag{7}$$

$$R = 287.102 \text{ J/(kg} \cdot \text{K)} \tag{8}$$

Then, $\gamma = 1.402$ and $C_P = 1001.29$ J/(kg · K) for a perfect gas [5].

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Table 1 Coefficients of the polynomial $C_P(T)$

i	a_i
1	1001.1058
2	0.04066128
3	−0.000633769
4	2.747475×10^{-6}
5	-4.033845×10^{-9}
6	3.069773×10^{-12}
7	$-1.350935 \times 10^{-15}$
8	3.472262×10^{-19}
9	$-4.846753 \times 10^{-23}$
10	2.841187×10^{-27}

The *chosen interpolation* of the C_P values according to the temperature is presented by relation (9) in the form of Horner scheme by

$$C_P(T) = a_1 + T\{a_2 + T[a_3 + T(a_4 + T\{a_5 + T[a_6 + T(a_7 + T\{a_8 + T[a_9 + T(a_{10})]\})]\})]\} \tag{9}$$

The constants ($a_i i = 1, 2, \ldots, 10$) of interpolation are illustrated in Table 1.

The relation (9) gives undulated dependence for temperature approximately lower than $\bar{T} = 240$ K. So for this field, the table value [5] was taken

$$\bar{C}_P = C_P(\bar{T}) = 1001.15868 \text{ J/(kg} \cdot \text{K)}$$

Thus, for $T \leq \bar{T}$, then $C_P(T) = \bar{C}_P$ and for $T > \bar{T}$, relation (9) is used.

The selected interpolation gives an error less than $\varepsilon = 10^{-3}$ between the table and interpolated values.

Once the interpolation is made, the function $H(T)$ of the relation (7) can be determined, by integration of the function $C_P(T)$ in the interval $[T, T_0]$. Then, $H(T)$ is a function with a parameter T_0 and it is defined when $T \leq T_0$.

When $M = 1.0$, then $\nu = 0.0$. The value of ν for $M > 1.0$ ($T < T_*$) is given by

$$\nu(T) = \int_T^{T_*} F_\nu(T)\,\mathrm{d}T \tag{10}$$

The PM function of the PG model [1] is explicitly connected to the Mach number, which represents the basic variable for this model. On the contrary for our model, the basic variable is the temperature, because of Eq. (4) connecting M and T, which we cannot determine an analytical expression of its reverse.

III. Calculation Procedure

Let us note that the function ν depends on T_0. Figure 1 presents its variation for aim to carrying out the suitable integration quadrature. The tracing is selected for $T_0 = 500$ K and $M_S = 6.00$.

One can show that the integration quadrature with constant step requires a very high discretization to have a good precision, considering the high gradient at the ends of the interval. A nodes condensation procedure is thus necessary in the vicinity of the temperatures T_* and T_S, for aim to calculate the value of the integral with high precision in a reduced time with a minimum nodes number of quadrature. The selected numerical integration quadrature is that of Simpson, presented in [8].

The condensation form chosen in our calculation is that of Robert, presented in [9] given by

$$s_i = b_1 z_i + (1 - b_1)\left[1 - \frac{\tanh[b_2(1 - z_i)]}{\tanh(b_2)}\right] \tag{11}$$

where

$$z_i = \frac{i-1}{N-1} \qquad 1 \leq i \leq N \tag{12}$$

The temperature distribution in the points of integration are given by

$$T_i = s_i(T_* - T_S) + T_S \tag{13}$$

It is necessary to condense the nodes towards the two ends at the same times considering the function has a high gradient in these points. Equation (11) can condense the nodes towards only one end. It is then necessary to divide the interval $[T_S, T_*]$ into two parts, (equal, for example) $[T_S, T_M]$ and $[T_M, T_*]$ such as

$$T_M = (T_S + T_*)/2 \tag{14}$$

For interval $[T_S, T_M]$, the value of b_1 near to zero can be taken, and for interval $[T_M, T_*]$, the value of b_1 near to 2.00 should be taken, respectively, to condense the nodes towards the left and the right end. The number N of points must be divided again into two parts.

The value of T_* and T_S corresponding, respectively, to $M = 1$ and $M = M_S$ can each one be obtained by the resolution of Eq. (4) by using the bipartition algorithm [5,8]. The two values depends on T_0. The value of ν_S can be obtained by integration of the function (3) in the interval $[T_S, T_*]$ by using the Simpson's quadrature with condensation of nodes towards the two ends.

According to this study, a difference between the results given by the two models is presented. For each values of (T_0, M_S), the relative error can be evaluated by the following relation:

$$\varepsilon_\nu\% = |1 - \nu_{\mathrm{PG}}/\nu_{\mathrm{HT}}| \times 100 \tag{15}$$

IV. Results and Comments

The results are to be presented only for air. Figure 2 presents the variation of the PM function according to the Mach number, for some values of T_0 including the PG case. If the variation of $C_P(T)$ is taken into account, the stagnation temperature influences the size of this function. The four curves are almost confounded until approximately $M = 2.00$, which is interpreted by the possibility to use the PG model as long as $M < 2.00$. The curves 3 and 4 are almost confounded some is the Mach number, which indicates that the use of the PG model gives acceptable results as long as T_0 is lower than 1000 K.

Figure 3 presents the variation of the PM function at high temperature according to T_0 when the Mach number $M = 3.00$. At low temperature, until approximately 240 K, the gas can be regarded as calorically perfect with an error $\varepsilon = 0.0$, considering in this interval, the function $C_P(T)$ is constant [3]. The more T_0 increases, the value of ν increases and moves away considerably from the PG

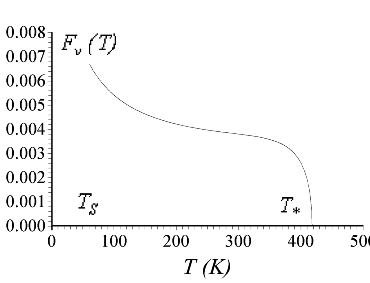

Fig. 1 Variation of the function $F_\nu(T)$ in $[T_S, T_*]$.

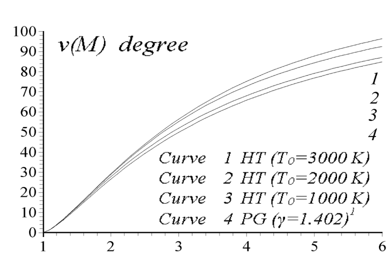

Fig. 2 Variation of $\nu(M)$ vs M.

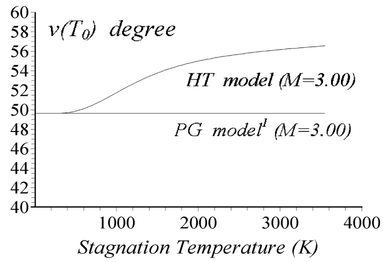

Fig. 3 Variation of ν vs T_0 when $M = 3.0$.

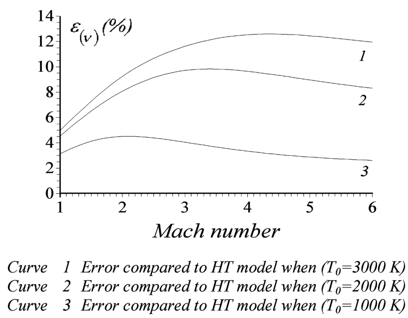

Fig. 4 Variation of the relative error given by the function ν of PG model vs Mach number.

model, from where one needs to use the HT model to correct the results.

Figure 4 presents the variation of the relative error given by the PM function of the PG model compared to the HT model for various values of T_0. The error varies with T_0 and the Mach number. For example, if $T_0 = 2000$ K and $M = 3.00$, the error is equal to $\varepsilon = 9.70\%$.

Numerical calculation shows that there is a difference in spite of low temperature. For example, when $T_0 = 298.15$ K and $M = 3.00$, we obtain [1] $\nu = 49.648$ deg. By comparison to that of the PG model $\nu = 49.651$ deg, an error $\varepsilon = 0.006\%$ is obtained.

Considering the PM function is equal to zero when $M = 1$, the error analysis has problem of calculation (zero under zero). Then, the obtained error is equal to

$$\varepsilon_\nu(M=1) = \lim_{M\to 1(T\to T_*)} \left|1 - \frac{\nu_{\rm PG}(M)}{\nu_{\rm HT}(T)}\right| \times 100 = \left|1 - \frac{0}{0}\right| \times 100$$
$$= \begin{cases} 3.104\% & \text{when } T_0 = 1000 \text{ K} \\ 4.510\% & \text{when } T_0 = 2000 \text{ K} \\ 4.949\% & \text{when } T_0 = 3000 \text{ K} \end{cases} \tag{16}$$

V. Conclusions

For an error lower than 5%, a study on the supersonic flow is presented by using the PG relations, if $T_0 < 1000$ K for any value of Mach number, or when $M < 2.00$ for any value of T_0 up to approximately 3000 K. The PG model is presented by a simple and explicit relation, and does not require high time to make calculation, unlike the proposed model, which requires the resolution of a nonlinear algebraic Eq. (4) and integration of a complex analytical function (3). It takes more time for calculation and data processing.

The presented relation is valid for any interpolation form chosen for the function $C_P(T)$. The essential is that the selected one gives small error.

Another substance instead of air can be chosen. It is necessary to have its table of variation of C_P with the temperature and to make a suitable interpolation.

The PG model relations can be obtained starting from the HT model ones by annulling all constants of interpolation except the first. It becomes a particular case of the HT model.

The PM function can be used to solve problems of external flows, in particular around a supersonic pointed airfoil. It leads us to develop a new mathematical model of the Method of Characteristics to concept and design various supersonic nozzles shapes at high temperature.

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